J Glob Optim (2007) 38:163–179
DOI 10.1007/s10898-006-9065-5

ORIGINAL PAPER

Extremal problems for convex polygons

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Received: 30 December 2005 / Accepted: 1 July 2006 / Published online: 9 August 2006

Abstract Consider a convex polygon V_n with n sides, perimeter P_n, diameter D_n, area A_n, sum of distances between vertices S_n and width W_n. Minimizing or maximizing any of these quantities while fixing another defines 10 pairs of extremal polygon problems (one of which usually has a trivial solution or no solution at all). We survey research on these problems, which uses geometrical reasoning increasingly complemented by global optimization methods. Numerous open problems are mentioned, as well as series of test problems for global optimization and non-linear programming codes.

Keywords Polygon · Perimeter · Diameter · Area · Sum of distances · Width · Isoperimeter problem · Isodiametric problem

1 Introduction

Plane geometry is replete with extremal problems, many of which are described in the book of Croft et al. [12] on *Unsolved problems in geometry*. Traditionally, such problems have been solved, some since the Greeks, by geometrical reasoning. In the last four decades, this approach has been increasingly complemented by global optimization methods. This allowed solution of larger instances than could be solved by any one of these two approaches alone.

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Probably the best known type of such problems are circle packing ones: given a geometrical form such as a unit square, a unit-side triangle or a unit-diameter circle, find the maximum radius, and configuration of n circles which can be packed in its interior (see [47] for a recent survey and the site [45] for a census of exact and approximate results with up to 300 circles).

Extremal problems on convex polygons have also attracted attention of both geometers and optimizers (e.g., see [19]). In the present paper, we survey research on that topic.

A *polygon* V_n is a closed plane figure with n sides. A *vertex* of V_n is a point at which two sides meet. If any line segment joining two points of V_n is entirely within V_n then V_n is *convex*. If all sides of V_n have equal length, V_n is *equilateral*. If an equilateral polygon has equal inner angles between adjacent sides it is *regular*. The *perimeter* P_n of V_n is the sum of the length of its sides. The *diameter* of V_n is the maximum distance between two points of V_n, or, which is equivalent, the length of its longest *diagonal*, a straight line joining two vertices. The *width* of V_n in direction θ is the distance between two parallel lines perpendicular to θ and supporting V_n from below and above. The *width* of a polygon V_n is the minimum width for all directions θ.

Two convex polygons V_n and V'_n are *isoperimetric* if they have the same perimeter. They are *isodiametric* if they have the same diameter. Isoperimetric problems for convex polygons consist in finding the extremal polygons for some quantity such as area, diameter, sum of distances or width while keeping the perimeter fixed (say at 1). Isodiametric problems are defined similarly.

Consider a convex polygon V_n with n sides. Let A_n denote its area, P_n its perimeter, D_n its diameter, S_n the sum of distances between all pairs of its vertices and W_n its width. Maximizing and minimizing any of these quantities while fixing another one defines 10 pairs of extremal problems. Usually, one problem from each pair has a trivial solution or no solution at all. To illustrate, maximizing A_n while fixing P_n is the classical isoperimetric problem for polygons whose solution is the regular polygon, whereas minimizing A_n while fixing P_n has as solution a polygon as close as desired from a straight line of length $P_n/2$ and its area tends to 0 in the limit. Additional constraints may be imposed, the most studied of which being that the polygons are equilateral.

At first thought, one might believe that the regular polygons are the solution of the extremal problems defined above. Indeed, this is often true, but usually only for some large subset of the values of n.

Table 1 specifies for which values of n these extremal problem have been solved. Values above the main diagonal correspond to general convex polygons and values below to equilateral polygons.

A simple formulation for these problems is obtained by denoting the consecutive vertices of the $n-$gon V_n by $v_i = (x_i, y_i)$:
$A_n = \left|\frac{1}{2}\sum_{i=1}^{n}(y_{i+1} - y_i)(x_{i+1} + x_i)\right|$, $P_n = \sum_{i=1}^{n}\|v_{i+1} - v_i\|$, $D_n = \max_{i<j}\|v_i - v_j\|$, $S_n = \sum_{i<j}\|v_i - v_j\|$, $W_n = \min_i \max_{j\neq i,i+1} \frac{|(y_{j+1}-y_j)x_i+(x_j-x_{j+1})y_i+x_{j+1}y_j-x_jy_{j+1}|}{\|v_{j+1}-v_j\|}$, $i+1$ and $j+1$ are done modulo n. Other formulations are used in the literature.

The first line of Table 1 corresponds to isoperimetric problems. They will be studied in Sect. 2. The second line corresponds to isodiametric problems. They will be examined in Sect. 3. Results below the main diagonal will be considered in those sections too. In Sect. 4, further open problems will be mentioned. We hope by this survey to

Table 1 Values of n for which extremal polygons have been determined

	P_n	D_n	A_n	S_n	W_n
$P_n = 1$	–	Min n with odd factor and $n = 4, 8$	Max all n	Open	Open
$D_n = 1$	Max n with odd factor and $n = 4, 8$	–	Max odd n	Max $n = 3, 4, 5$	Max n with odd factor and $n = 4$
$A_n = 1$	Min all n	Min odd n	–	Open	Open
$S_n = 1$	Open	Min $n = 3, 5$	Open	–	Open
$W_n = 1$	Open	Min n with odd factor and $n = 4$	Open	Open	–

make geometers more aware of the help that global optimization can bring them, to stimulate work of optimizers in applying global optimization to plane geometry as well as to provide sets of test problems for both exact and heuristic algorithms from that field (including methods of non-linear programming).

2 Isoperimetric problems

2.1 Maximizing the area

The oldest isoperimetric problem for convex polygons dates from the Greeks and can be expressed as follow: *Find for all n which convex polygon V_n with unit perimeter has maximum area*. As explained in a beautiful paper of Blåsjö [9], this problem was solved (assuming implicitly the existence of a solution) by Zenodorus (circa 200–140 b.c.) in his lost treatise “On isometric figures”. Happily, his work was reported by Pappus [49] and Theon of Alexandria [22]. The solution is the set of regular polygons. The argument comprises two steps: (1) show that among all isoperimetric triangles with the same base the isosceles triangles has maximum area. It follows, by iteration, that the optimal polygon must be equilateral; (2) show that if the polygon is not equiangular, its area may be increased by redistributing perimeter from a pointy to a blunt angle until the two angles are the same. Increasing the number of sides gives in the limit the famous result that among all plane figures with fixed perimeter the circle encloses the largest area.

2.2 Minimizing the diameter

A second isoperimeteric problem has apparently not been considered until a paper of Reinhardt [38] published in 1922, but has been much studied since, both for general and for equilateral polygons. It can be expressed as follows: *Find for all n which polygon with unit perimeter has the smallest diameter*.

As mentionned in the introduction this problem is equivalent to finding the polygons with unit diameter (which, following Graham [21] we call *small* polygons) and longest perimeter. As it is usually considered in this second form in the literature, we will study it in the next section, devoted to isodiametric problems.

2.3 Minimizing and maximizing the sum of distances between vertices

These problems appear to be unsolved. They can be stated as follows: *find for all n which polygons with unit perimeter has the smallest, or largest, sum of distances between all pairs of vertices*.

Numerical experiments suggest in both cases that extremal polygons appear to be arbitrarily close to the straight line of length $\frac{1}{2}$, all vertices being partitioned at the endpoints of the line segment. We therefore make the following conjecture.

CONJECTURE 2.1 *The sum S_n of distances between vertices $v_1, v_2, \ldots, v_n$ of a convex polygon with unit perimeter satisfies*

$$\frac{n-1}{2} < S_n < \frac{1}{2}\left\lfloor\frac{n}{2}\right\rfloor\left\lceil\frac{n}{2}\right\rceil$$

the bounds being approached arbitrarily closely by the line segment $[0, \frac{1}{2}]$ *with* v_1 *at* 0, *and* $v_2, \ldots, v_n$ *arbitrarily close to* $\frac{1}{2}$ *in the former case, and with* $v_1, v_2, \ldots, v_{\lfloor \frac{n}{2} \rfloor}$ *arbitrarily close to 0 and* $v_{\lfloor \frac{n}{2} \rfloor+1}, \ldots, v_n$ *arbitrarily close to* $\frac{1}{2}$ *in the latter case.*

2.4 Maximizing the width

This problem also appears to be unexplored. It is expressed as: *find for all n which polygon with unit perimeter has the largest width*.

The optimal figure for $n = 4$ is not the square since its width $\frac{1}{4}$ is inferior to the width of the equilateral triangle $\frac{\sqrt{3}}{6}$. The best solution that we have so far has a width of $\frac{1}{4}\sqrt{(6\sqrt{3}-9)}$.

3 Isodiametric problems

3.1 Maximizing the area

3.1.1 Exact solution

As for the small polygon with longest perimeter, main results on the small polygons with maximum area were given by Reinhardt [38]. He proves that for all odd $n \geq 3$, the regular polygon has maximum area among all polygons with unit diameter. He also observes that the square has maximum area among small quadrilaterals, but the solution is not unique. Indeed, any small quadrilateral whose diagonals are of unit length and perpendicular has an area equal to $\frac{1}{2}$.

As recently noted by Mossinghoff [31, 32], Reinhardt also states, at the end of his paper, the often rediscovered result that the regular small polygon never has maximum surface for even $n \geq 6$ (see [26, 40, 48]).

Graham [21] determined the small hexagon with largest area in 1975. In order to do so, he considered all *diameter configurations*. They can be represented by *diameter graphs* with rigid edges of length 1. As shown by Erdős [16] and Woodall [54] such graphs have the following properties:

(a) Any two edges have a point in common, which is either an endpoint or an interior point for both of them;

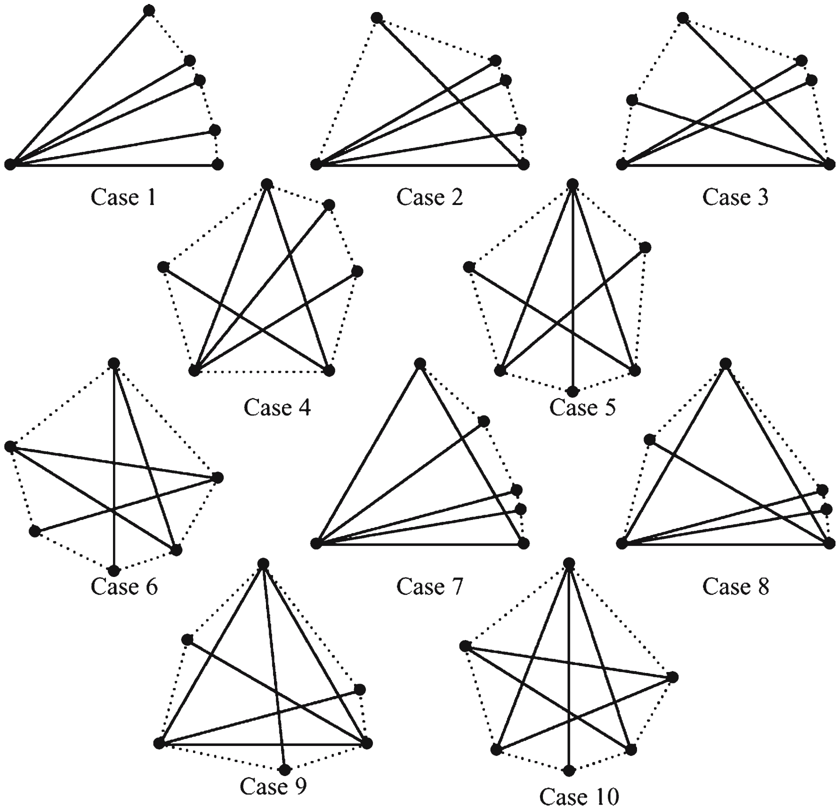


Fig. 1 The ten possible diameter configurations for the hexagon

(b) If the diameter graph is a tree, i.e., it is connected and has no cycle, it is a *caterpillar*, i.e., a path to which are incident *pending edges* (i.e., edges with the other vertex of degree 1) only;

(c) If the diameter graph is connected but not a tree, it has a single cycle, of odd length to which are incident all other edges, which are pending ones.

Enumeration shows that there are 10 different connected diameter graphs for the hexagon. They are reproduced on Fig. 1. Using geometric arguments, Graham eliminates all but the last one. For instance, the area of an hexagon with diameter configuration 1 cannot exceed the area of a sector with radius 1 and angle $\frac{\pi}{3}$, which is $\frac{\pi}{6} \approx 0.523599\ldots$ This maximum area is less than the area of the regular small hexagon, i.e., $\frac{3\sqrt{3}}{8} \approx 0.649519\ldots$

Graham then proves (without giving details; a full proof can be found in [55]) that the optimal hexagon has an axial symmetry. Then the following expression for the area, in a single variable, can be derived:

$$A_6 = \left(\frac{1}{2} - x\right)\left(1 - x^2\right)^{1/2} + x\left(1 + \left(1 - \left(x + \frac{1}{2}\right)^2\right)^{1/2}\right).$$

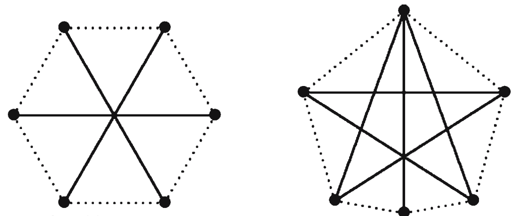
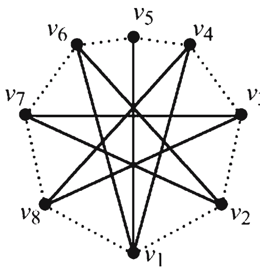


Fig. 2 Small hexagons with maximal area



$$
\begin{array}{ll}
\max\limits_{v_i=(x_i,y_i)} & A_8 = \frac{1}{2}\sum_{i=1}^{8}(y_{i+1}-y_i)(x_{i+1}+x_i) \\
\text{s.t.} & \|v_i - v_{i+3}\|^2 = 1, i \in \{1,6,7,8\} \\
 & \|v_i - v_{i+4}\|^2 = 1, i \in \{2,3,4\} \\
 & \|v_1 - v_5\|^2 = 1 \\
 & \|v_i - v_j\|^2 \leq 1, i < j \in \{1,\ldots,8\}.
\end{array}
$$

Fig. 3 Definition of variables of case 31, optimal according to the conjecture of Graham

This expression has a unique maximum for $x \in [0, \frac{1}{2}]$, which can be obtained by setting the first derivative at 0 and solving numerically. The largest small hexagon is represented in Fig. 2, together with the regular small hexagon. Its area is $A_6^* \approx 0.674981\ldots$ that is about 3.92% larger than that of the regular small hexagon. Note that Graham's hexagon had previously been found by Bieri [8] under an unproved assumption of symmetry.

The approach of Graham has been followed by the present authors, together with Xiong [6], to determine the largest small octagon. There are more cases to consider than for the hexagon, i.e., 31 diameter configurations. The optimal diameter configuration, already conjectured by Graham [21], and the problem of finding the corresponding octagon with largest area expressed as a nonconvex quadratic program with nonconvex constraints are illustrated in Fig. 3.

The objective function of the quadratic program, represents the area of the octagon. The constraints express that the (squared) distances between vertices do not exceed 1, and that distances corresponding to diameters are equal to 1. The formulation requires 10 variables, 10 linear inequalities, 17 quadratic inequalities, and 6 quadratic equalities. There were two erroneous signs in the problem formulation in [1, 6]. A corrected version may be found in [4].

To determine the optimal solution, the algorithm of Audet et al. [2] was used. It is a branch-and-cut method based on the Reformulation-Linearization-Technique (RLT, see [41–43]). Squares and products of pairs of variables are replaced by new variables and linear constraints added to ensure having a best possible approximation. Branching is done in such a way that the local error for a term is as small as possible. Moreover, cuts due to branching may be used at different nodes of the branch-and-bound tree. Finally, resolution begins by the determination of intervals of feasible values for each variable, which notably strengthens bounds and leads to more efficient branching.

Computing time for the case 31 was over 100 h on a SUN-SPARC 20 station [1]. The largest small octagon is represented in Fig. 3. Its area is $A_8^* = 0.726867\ldots$ and is about 2.79% larger than $\frac{\sqrt{2}}{2} = 0.707107\ldots$, the area of the small regular octagon. To complete the proof it was required to eliminate the 30 remaining diameter configurations. This was done by combining geometric reasoning (which suffices in 23 cases) with numerical methods (in the remaining seven cases).

3.1.2 Bounds for polygons with many sides

Instead of determining exact values of the area or perimeter for the remaining even cases, Mossinghoff [31, 32] follows a different strategy: he constructs approximate solutions with a guarantee on the value, i.e., an upper bound on the error expressed as a function of the order $O(n^s)$ of the sth power of the number of vertices. He first notes that it is well-known from the solution of the isoperimeteric problem for polygons that

$$A_n \le \frac{P_n^2}{4n} \cot\left(\frac{\pi}{n}\right).$$

Combining this result with Reinhardt's bound [38] for the perimeter, i.e., $P_n \le 2n\sin(\frac{\pi}{2n})$, gives for small polygons

$$A_n \le \frac{1}{2}\cos\left(\frac{\pi}{n}\right)\tan\left(\frac{\pi}{2n}\right) = M_n$$

an upper bound on the area. Then, Mossinghoff considers the diameter configuration conjectured to be optimal by Graham [21]. With some simplifiying assumptions related to equality of smallest angles of the star polygon, he then computes exact values for the 2 or 3 remaining ones. This leads to a family of small polygons A'_n such that, when n is an even integer:

$$A'_n - A^r_n = \frac{\pi^3}{16n^2} + \mathrm{O}\left(\frac{1}{n^3}\right) \quad \text{and} \quad M_n - A'_n < \frac{2\pi^3}{17n^3} + \mathrm{O}\left(\frac{1}{n^4}\right),$$

where A^r_n denotes the regular small polygon with n sides. Hence the error decreases with the inverse of the third power of n.

3.1.3 Approximate solutions

In addition to exact results and bounds, approximate results (or unproved optimal ones) have been obtained both by metaheuristics and by nonlinear programming. Schilbach [44] presents a simple Monte–Carlo method to find a polygon with approximately maximum area. His applet works $n = 6$ to about $n = 20$. Rechenberg [37] applies an evolutionary strategy to determine this maximum area polygon with $n = 6, 8$, and 10. For $n = 6$, he obtains an hexagon with an area 3.94% above that of the regular one and for $n = 8$ an octagon with an area 2.79% above that of the regular octagon. These values are close to the optimal ones (in fact, slightly above them due probably to tolerances in constraint satisfaction).

The maximum area problem has also been extensively used in the comparison of non-linear programming codes. The formulation adapted relies on polar coordinates.

Taking (r_i, θ_i) for $i = 1, \ldots, n-1$ and $r_n = 0, \theta_n = \pi$ as the coordinates of the vertices of V_n, this problem can be expressed as

$$\begin{array}{lll} \max\limits_{r,\theta} & \text{area}(r,\theta) = \dfrac{1}{2}\sum\limits_{i=1}^{n-1} r_{i+1} r_i \sin(\theta_{i+1} - \theta_i) & \\ \text{s.t.} & r_i^2 + r_j^2 - 2r_i r_j \cos(\theta_i - \theta_j) \le 1, & 1 \le i < j \le n-1, \\ & \theta_i \le \theta_{i+1}, & 1 \le i \le n-2, \\ & 0 \le \theta_i \le \pi, & 1 \le i \le n-1, \\ & 0 \le r_i \le 1, & 1 \le i \le n-1. \end{array} \tag{1}$$

The problem has $2n-2$ variables, $\left(\frac{n}{2}+1\right)(n-1)+1$ general constraints of which $n-2$ are linear, $2n-2$ bound constraints and a large number of local optima. The objective function is equal to the area, decomposed into triangles with a common vertex. The first $\frac{(n-2)(n-1)}{2}$ constraints express that the distance between any 2 of the first $n-1$ vertices does not exceed 1. The next constraints impose, without loss of generality, an order and a maximum value on the angle coordinates. The last constraints express that the distances between v_n and another vertex v_i cannot exceed 1. Values of n considered are $6, 10, 20, 25, 50, 75, 100$, and 200. Results are given in a series of reports by Moré et al. [10, 14, 15]. They are brought together in Table 2 (when several different values are reported, the last best one is kept, stars indicate no convergence in the allowed time). Initial solutions were "a polygon will almost equal sides" [10]. Codes considered are DONLP2 [46], LANCELOT [11], MINOS [33], SNOPT [20], LOQO [50], FILTER [18], and KNITRO [52]. In Table 2, the marks –, ? and ⋆ indicate, respectively, that this test is not performed, that the CPU-time is not given and that the algorithm used for this test does not converge.

In addition, we recall in the bottom lines of the table the area A_n^r of the regular polygon with n sides, A_{n-1}^r of the regular polygon with $n-1$ sides (which is larger

Table 2 Performance on largest small polygon problem (from Moré et al. [10, 14, 15]

Nb Var. Solver	$n=6$	$n=10$	$n=20$	$n=25$	$n=50$	$n=75$	$n=100$	$n=200$
DONLP2	0.674979	0.749136	0.776853	–	⋆	–	⋆	–
CPU time	?	?	?	–	⋆	–	⋆	–
LANCELOT	0.674982	0.749137	0.776859	0.779715	0.783677	0.784747	0.785031	–
CPU time	?	?	?	12.83 s	398.12 s	1148.2 s	?	–
MINOS	0.674981	0.749137	0.768788	0.764383	0.766297	0.760729	0.762473	0.657163
CPU time	?	?	?	2.11 s	5.6 s	98.54	?	223.94 s
SNOPT	0.674981	0.749137	0.776859	0.77974	0.784015	0.784769	0.785056	⋆
CPU time	?	?	?	1.14 s	4.34 s	91.07 s	(256.6 s)?	⋆
LOQO	0.674981	⋆	0.719741	0.76714	0.77520	0.777554	⋆	⋆
CPU time	?	⋆	?	3.49 s	3.16 s	268.9 s	⋆	⋆
FILTER	–	–	–	–	0.766131	–	0.777239	⋆
CPU time	–	–	–	–	27.64 s	–	555.2 s	⋆
KNITRO	–	–	–	–	0.760725	–	0.737119	0.674980
CPU time	–	–	–	–	1.41 s	–	8.99 s	59.53 s
A_n^r	0.649519	0.734732	0.772542	0.780232	0.783333	0.784824	0.784881	0.785269
A_{n-1}^r	0.657164	0.745619	0.776456		0.784053		0.785069	0.785316
A_{n-1}^{cor}	0.672288	0.748257	0.776738		0.784070		0.785071	0.785317

than A_n^r for $k \geq 6$) as well as A_{n-1}^{cor} of the regular polygon with $n-1$ sides corrected by adding a vertex at distance 1 along the mediatrix of an angle.

Results for small n appear to be good: for $n = 6$, the five codes give the optimal value of 0.67498; four of them obtain the same presumably optimal value of 0.74914 for $n = 10$ (the fifth code did not converge), two of them get the same value of 0.77686 for $n = 20$ (the fourth gets a slightly lower value and the fifth a much lower one).

For larger values of n, results are not so good, and sometimes quite bad. None of the codes obtains the optimal solution for $n = 25$ or $n = 75$ nor a solution of value better than A_{n-1}^{cor} for $n = 50, 100$ or 200. Both LANCELOT and SNOPT provide values close to the best ones for $n \leq 100$, SNOPT being the fastest to do so. The other codes get worse values, sometimes by a very wide margin. For $n = 200$ the solutions reported for the two codes successful in generating a solution are 16% and 14% below the area of the regular polygon A_{200}^r. It appears in that case, and probably in others, that these codes return a worse solution than the initial one.

So, these tests problems are good ones in the sense that they are challenging and discriminating. Further comparisons could be made when beginning with a solution farther away from the optimal one, or considering a particular case defined by a diameter configuration of, as done in Mossinghoff [31, 32]. The other problems considered in this survey could also lead to sets of test problems, both for nonlinear programming codes and for metaheuristics using them as descent or ascent routines.

3.2 Maximizing the perimeter

This problem, already mentioned in its equivalent form in the previous section, may be expressed as follows: *Find for all n which unit diameter polygon has the largest perimeter*.

3.2.1 Reuleaux polygons

The problems of maximum perimeter will bring us to consider a class of geometric figures already studied by Reuleaux [39] about a century ago. *Reuleaux polygons* are not, strictly speaking, polygons, but have a polygonal basis, i.e., an odd polygon, regular or not, with the property that each vertex is at distance 1 from the two vertices of the opposite side. The Reuleaux polygon is obtained by replacing each side by an arc circle with radius 1, centered at a vertex and joining the two vertices of the opposite side. Three examples of Reuleaux polygons, regular or not, are represented in Fig. 4.

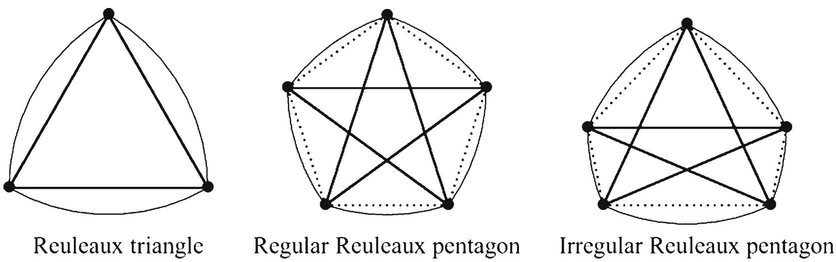


Fig. 4 Examples of reuleaux polygons

A remarkable property of Reuleaux polygons is that their width is constant, i.e., the same in all directions. It is also worth noting that the three unit-diameter Reuleaux polygons of Fig. 4 have a perimeter equal to π. This is a consequence of Barbier's theorem: "All curves of constant width w have the same perimeter πw". Moreover, this value is equal to the sum of internal angles of the polygon, or star polygon, formed by the diameters. Consequently, the problem of finding the convex set with unit diameter and longest perimeter has as solution the circle, but also all unit-diameter Reuleaux polygons.

3.2.2 A bound and some easy cases

Reinhardt [38] gave an upper bound of $2n \sin \frac{\pi}{2n}$ on the perimeter of a small polygon for all $n \neq 2^s$ where s is a positive integer. Datta [13] showed that it also holds when n is a power of 2. This bound is attained for regular polygons when n is odd. So regular small polygons have maximum perimeter for odd n. Moreover, if n is even but not a power of 2, i.e., $n = m2^s$ where s is an odd number, prime or not, an optimal polygon can be built from a Reuleaux polygon as follows:

(a) Consider a regular small polygon with m sides;
(b) Transform this polygon into a Reuleaux polygon by replacing each edge by a circle's arc passing through its end vertices and centered at the opposite vertex;
(c) Add at regular intervals, $2^s - 1$ vertices within each edge;
(d) Take the convex hull of the vertex set, i.e., vertices of the Reuleaux polygon and vertices added in c).

Then Reinhardt's bound is attained. Indeed each angle of the star polygon inside the regular small polygon is $\alpha = \frac{\pi}{m}$ and each of them is divided by 2^s upon addition of the new vertices. So $2 \sin \frac{\alpha/2^s}{2} = 2 \sin \frac{\pi/m2^s}{2} = 2 \sin \frac{\pi}{2n}$. Examples of small polygons with maximum perimeter and $n = 6$, 10 and 12 sides are given in Fig. 5. Observe that if m is a composite number, there are several equivalent solutions to the problem of the small polygon with maximum perimeter. Datta [13] shows that all of them can be determined by solving a system of diophantine equations. An example with three optimal solutions is the case $n = 15$ illustrated in Fig. 6.

3.2.3 Bounds for polygon with many sides

Tamvakis [48] considered a Reuleaux triangle with unit width, with one arc divided into either $\lfloor \frac{n}{3} \rfloor$ or $\lceil \frac{n}{3} \rceil$ equal subarcs, so that the total number of vertices (at endpoints

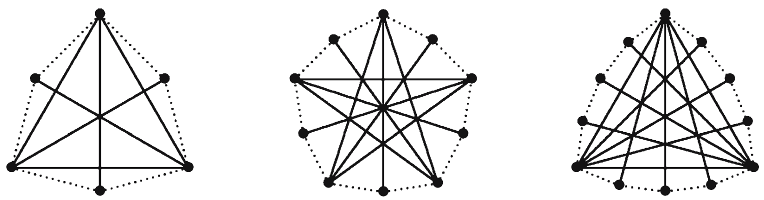


Fig. 5 Polygons with maximal perimeter with even n and $n \neq 2^s$

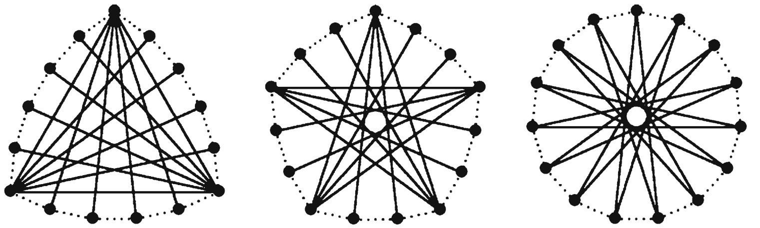


Fig. 6 Three pentadecagon of maximal perimeter $P_n = 30\sin(\frac{\pi}{30}) \approx 3.135854\ldots$

of arcs or subarcs) is equal to n. He asks if these polygons with perimeter P_n^T are optimal for $n = 2^s$ and $s \geq 3$. The maximum perimeter octagon described in subsection 3.2.4 shows it is not the case. Indeed its perimeter is $P_n^* \approx 3.121147\ldots$ while that the Tamvakis polygon is $P_n = 12\sin(\frac{\pi}{18}) + 4\sin(\frac{\pi}{12}) \approx 3.119154\ldots$.

Moreover, using similar reasoning as in the maximum area case, Mossinghoff [31, 32] found a family of small polygons with perimeter P_n^H such that

$$P_n^H - P_n^T = \frac{\pi^3}{4n^4} + \mathrm{O}\left(\frac{1}{n^5}\right) \quad \text{and} \quad R_n - P_n^H = \frac{\pi^5}{16n^5} + \mathrm{O}\left(\frac{1}{n^6}\right),$$

where R_n denotes the value of the Reuleaux bound on the perimeter. So this time the error decreases with the inverse of the fifth power of n.

3.2.4 Exact solutions for $n = 4$ and $n = 8$: general polygons

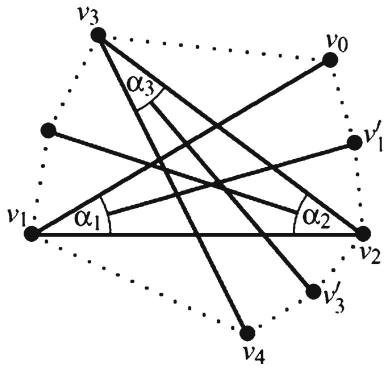
The general small quadrilateral with largest perimeter has been determined by Tamvakis [48]. Three of its vertices are those of a unit equilateral triangle, and the fourth one is at unit distance from one of the three vertices and equidistant from the other two. Its perimeter is equal to $2(1+\sqrt{2-\sqrt{3}}) = 2+4\sin\frac{\pi}{12} \approx 3.035276$ which exceeds the perimeter of the small square $2\sqrt{2} \simeq 2.828427$ by about 7.3%. Datta [13] also gives a proof of this result.

For $n = 8$, we have solved the problem of finding the small octagon both in the general case [3] and for equilateral octagons (see next subsection). For the former, we first proved that the optimal diameter configuration graph of the octagon is connected, so it is one of the 31 configurations of diameters considered previously for the largest small octagon. This time, it is the diameter configuration 29 which gives the optimal solution. This solution was obtained by solving the case of configuration 10, which is a relaxation of configuration 29, as shown on Fig. 7: the constraint $\|v_0 - v_4\| \leq 1$ in case 10 becomes $\|v_0 - v_4\| = 1$ in case 29.

The non-convex program corresponding to this case can be written:

$$\begin{aligned} \max_{\alpha}\ & 4\sin(\tfrac{\alpha_1}{4}) + 4\sin(\tfrac{\alpha_2}{4}) + 4\sin(\tfrac{\alpha_3}{4}) + ||v_1 - v_4|| + ||v_0 - v_3|| \\ \text{s.t.}\ & ||v_0 - v_4|| \leq 1 \\ & 0 \leq \alpha_i \leq \tfrac{\pi}{3}, \qquad i = 1,2,3, \end{aligned} \tag{2}$$

Fig. 7 Configuration 10



where the coordinates of vertices to be fixed in order to deduce the optimal solution are $v_0 = (\cos\alpha_1, \sin\alpha_1), v_1 = (0,0), v_2 = (1,0), v_3 = (1 - \cos\alpha_2, \sin\alpha_2)$ and $v_4 = (1 - \cos\alpha_2 + \cos(\alpha_2 + \alpha_3), \sin\alpha_2 - \sin(\alpha_2 + \alpha_3))$, see again Fig. 7.

Solving this problem with a branch-and-bound global optimization code using interval arithmetic [30, 36], bounds computed by the admissible simplex method [25, 29] and constraint propagation technique [27, 28], implemented in the IBBA code, we obtained the optimal solution in about 3 h of computing time on the 30 PC cluster of Pau University. Its perimeter is $P_8^* = 3.121147\ldots$, with an error not exceeding 10^{-6}. Analysis of this solution showed that the constraint $\|v_0 - v_4\| \leq 1$ is satisfied as an equality and thus this solution corresponds to configuration 29. Adding to problem (2) the first-order conditions

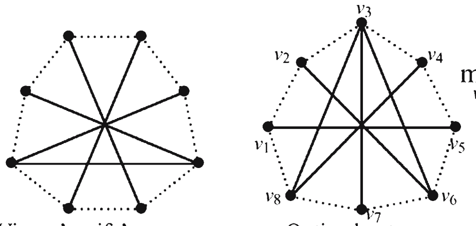
$$\frac{\partial\left(\|v_2 - v_1'\| + \|v_1' - v_0\| + \|v_0 - v_3\|\right)}{\partial\alpha_1} = 0,$$
$$\frac{\partial\left(\|v_2 - v_3'\| + \|v_3' - v_4\| + \|v_4 - v_1\|\right)}{\partial\alpha_3} = 0,$$

where $v_1' = \left(\cos(\frac{\alpha_1}{2}),\ \sin(\frac{\alpha_1}{2})\right)$ and $v_3' = \left(x_3 + \cos(\alpha_2 + \frac{\alpha_3}{2}),\ y_3 - \sin(\alpha_2 + \frac{\alpha_3}{2})\right)$, as well as the bound constraints $0.688 \leq \alpha_i \leq 0.881, \forall i \in \{1,2,3\}$, the branch-and-bound algorithm shows in only 0.12 s that case 10 can be eliminated. The remaining cases are solved by dominance arguments or numerically (see [3] for details).

3.2.5 Exact solutions for $n = 4$ and 8: equilateral polygons

It is easy to see that the square, i.e., the regular quadrilateral, is the small equilateral quadrilateral with largest perimeter. A similar property does not hold for $n = 8$. Indeed, Vincze [51] studies the maximum perimeter of equilateral polygons in a paper of 1950, and presents a small equilateral octagon, credited to his wife, which has a larger perimeter that the regular one. This octagon is represented in Fig. 8. In [5] we have shown, 54 years latter, that this octagon is suboptimal (see again Fig. 8).

This was done by assuming vertical symmetry and solving analytically with Maple for the diameter configurations of Fig. 8. To solve the problem completely, we proved among other results, that the four main diagonals of the optimal octagon must be diameters (as for the regular octagon). Then the quadratic program with non-convex quadratic constraints of Fig. 8 was solved. The objective function is just eight times the (unknown) length a of a side; the constraints express that all sides have the same



Vincze's wife's octagon Perimeter ≈ 3.0912...

Optimal octagon Perimeter≈ 3.095609...

$$\begin{aligned} \max_{v,a} \quad & 8a \\ \text{s.t.} \quad & \|v_i - v_{i+1}\|^2 = a \quad i=1,\dots,8 \\ & \|v_i - v_{i+4}\|^2 = 1 \quad i=1,2,3,4 \\ & \|v_i - v_j\|^2 \le 1 \quad i<j\in\{1,\dots,8\} \\ & \sin(\tfrac{\pi}{8}) \le a \le \tfrac{\pi}{8}. \end{aligned}$$

Fig. 8 Equilateral octagon of maximum perimeter

length a, that the main diagonals have length 1, that no pair of vertices are more than one unit apart. Upper and lower bounds on a are also derived. The RLT techniques [2] required only 45 s to find the optimal solution with an error not exceeding 10^{-7}.

3.3 Maximizing the width

A third isodiametric problem has attracted some attention. It can be described as follows: *Find for all n which convex polygon with unit diameter has the largest width*.

Bezdek and Fodor [7] studied this problem for general polygons. Results obtained are similar to those of the maximum perimeter problem:

(a) The maximum width satisfies $W_n \le \cos \frac{\pi}{2n}$ for $n \ge 3$ and equality holds if n has an odd divisor greater than 1;
(b) if n has an odd prime divisor then a polygon V_n is extremal if and only if it is equilateral and is inscribed in a Reuleaux polygon of constant width 1, so that the vertices of the Reuleaux polygon are also vertices of V_n;
(c) results are also obtained for quadrilaterals: $W_4^* = \frac{\sqrt{3}}{2}$. Moreover, all extreme quadrilateral have the property that 3 of their vertices from a regular triangle and the fourth vertex is contained in the Reuleaux triangle determined by the three vertices.

This last result implies that the maximum perimeter quadrilateral of Tamvakis [48] also has maximum width. No complete results are available for larger powers of 2 than 4.

3.4 Maximizing the sum of distances

The fourth isodiametric problem that we consider was studied by a few authors. It can be expressed as follows: *find for all n which convex polygon with unit diameter has the largest sum of distances between pairs of vertices*.

This problem appears to have been first posed by Fejes Tóth [17]. It may be stated mathematically as

$$\begin{aligned} \max_{v_1,v_2,\dots,v_n} S_n = \sum_{i,j=1}^{n} \|v_i - v_j\|, \\ \text{s.t. } \|v_i - v_j\| \le 1, \quad i,j = 1,2,\dots,n. \end{aligned} \qquad (3)$$

Wolf [53] obtained the following bound, valid also for arbitrary norms on $\mathbb{R}^2$:

$$\underset{n\geq 1, v_1, v_2, \ldots, v_n \in \mathbb{R}^2}{\text{Supremum}} \frac{1}{n^2} S_n \leq \frac{1}{2} + \frac{\pi}{16} \approx 0.6963495. \tag{4}$$

The problem was further studied by Pillichshammer [34, 35]. Instead of considering diameter configurations he followed another tack: pack triangles, quadrilaterals and pentagons in the complete graph. Then use upper bounds on the length of a triangle, quadrilateral or pentagon and keep an upper bound of 1 for all edges not covered. This leads to the following result [35]:

$$S_n \leq \begin{cases} \frac{n(n-1)}{3} + \frac{\pi(n^2+n-2)}{9} & \text{for } n = 1 \bmod 3, \\ \frac{n(n+1)}{3} + \frac{\pi(n^2-n-2)}{9} & \text{for } n = 2 \bmod 3, \\ \frac{n(n-3)}{3} + \frac{\pi(n^2+3n)}{9} & \text{for } n = 3 \bmod 3. \end{cases} \tag{5}$$

These bounds, divided by n^2, are shaper than the bounds (4): they tend to $\frac{1}{3} + \frac{\pi}{9} \approx 0.682391 < \frac{1}{2} + \frac{\pi}{16}$. The bounds (5) are sharp for $n = 3, 4, 5$, which implies that the triangle, the quadrilateral of Tamvakis [48] and the pentagon have minimum sum of distances between vertices.

Other bounds obtained by Pillichshammer in a similar way [35] involve trigonometric functions.

4 Conclusion and open problems

Work on extremal problems for polygons is substantial and rapidly increasing. Many problems remain open and even more unexplored. Indeed:

(1) Only one of the 10 problems for general polygons considered in Table 1 is solved for all values of n (max A_n subject to $P_n = 1$); two more problems are solved for all n having an odd factor and $n = 4$ or 8 (max P_n subject to $D_n = 1$) or $n = 4$ (max W_n subject to $D_n = 1$); one problem is solved for n odd and for $n = 4, 6, 8$ (max A_n subject to $D_n = 1$); one problem is solved for $n = 3, 4, 5$ only (max S_n subject to $D_n = 1$) and the five remaining problems are almost completely open;

(2) solved cases for equilateral polygons are the same as for general polygons except for a couple of cases ($n = 4$ for max S_n subject to $D_n = 1$ and max A_n subject to $S_n = 1$). So again many problems are open;

(3) extremal problems on polygons could also be expressed in terms of many quantities little or not studied up to now in that context. In particular the following appear to be promising:
   (a) The *circumradius* or radius of the smallest circle enclosing the polygon. It is known that given the lengths $a_1, a_2, \ldots, a_n$ of the successive sides of a polygon the maximum area is reached if all vertices are on a circle. It follows from that result and the argument of Zenodorus that the maximum area polygons with given circumradius are the regular ones. They also have maximum perimeter.
   (b) The *inradius* or radius of the largest circle enclosed the polygon.
   (c) The *fatness* of the polygon or ratio of the circumradius and inradius.

(d) The *length of a median*, i.e., the length of the line segment perpendicular to a side through its middle point and within the polygon. One could consider both the largest median length and the sum of median lengths.

(e) The *mediatrix length*, i.e., the length of the line segment passing through a vertex, dividing into the corresponding angle and within the polygon. Again, one could consider the largest mediatrix length and the sum of mediatrix lengths.

(4) Little work appears to have been done for extremal *star polygons* SP_n. They are obtained from convex polygons by joining vertices i and $(i+k) \bmod n$ along the perimeter for $i = 1, 2, \ldots, n$ and some $k = 2, 3, \ldots, \frac{n-1}{2}$. The *kernel* of a star polygon is another polygon equal to the intersection of the half planes limited by the star polygon's sides and including the center. It is also the set of points from which any point u of SP_n is *visible*, i.e., joined to u by a straight line entirely contained in SP_n. All problems studied above can be defined for the kernel (of order k) of a polygon in a similar way as done above for the polygon itself.

(5) Extension of problems from polygons to polytopes has been slightly studied; while formulating versions of the extremal problems discussed above for polytopes instead of for polygons is easy, their resolution appears to be difficult in view of the fact that results have been obtained up to now for particular cases or restricted families only, even in $\mathbb{R}^3$. For instance, Klein and Wessler [23, 24] determine the largest small three-dimensional polytope with six vertices and work on extending this result to $n+3$ vertices in n-dimensional space.

Acknowledgements Work of the first author was supported by NSERC grant 239436-01 and AFOSR grant F49620-01-1-0013. Work of the second author was supported by NSERC grant 105574-02.

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